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VARIOUS TOPOLOGICAL INDICES OF TOTAL GRAPHS OF LADDER AND WHEEL

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Abstract
Topological indices are graph invariants. In chemical graph theory, a molecule can be modeled by a graph by replacing atoms by the vertices and bonds by the edges of this graph. In this paper we study the Szeged index and Revised Szeged index of the total graph of a ladder graph and total graph of a wheel graph.

Introduction

A topological graph index, also called a molecular descriptor, is a mathematical formula that can be applied to any graph which models some molecular structure. In the graph-theoretic sense topological index is a graph invariant. The interest in topological indices is mainly related to their use in non-empirical quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QASR).
Throughout this paper we consider simple, finite, undirected and connected graph $G=\big(V(G),E(G)\big)$. The number of vertices of G is called order of G. We give a summary of definitions which are useful in this paper. For that we use “A first look at graph theory" by John Clark and Holton Derek Allan [1].

Definition 1. Capobianco M. and Molluzzo J. C. [2] introduced the concept of total graph. The total graph of graph $G=\big(V(G),E(G)\big)$ is denoted by $T(G)$. The $V\big(T(G)\big)=V(G)\cup E(G)$ and two vertices are adjacent in $T(G)$ if they are either adjacent in G or incident in G.
Definition 2. The ladder graph L_n is a cartesian product of path graph P_n and path graph p_2.
Definition 3. The wheel graph W_{n+1} is obtained by joining one apex vertex to every vertex of cycle C_n.
Definition 4. Distance between any pair (u,v) of vertices of graph $G=\big(V(G),E(G)\big)$ is denoted by $d(u,v)$ and is defined as the length of one of the shortest paths from u to v. Neighborhood of any vertex $u\in V(G)$ is set of all the vertices adjacent to u in G.
Definition 5. Let $G=\big(V(G),E(G)\big)$ be a graph. Diameter of G is denoted by $\operatorname{diam}(G)$ and is defined as $\operatorname{diam}(G)=\max_{u,v\in V(G)} d(u,v)$.
Definition 6. Let $G=\big(V(G),E(G)\big)$ be a graph and let $e=uv\in E(G)$. Consider following three sets: $N_1(e)=\{w\in V(G)\mid d(u,w)<d(v,w)\}$; $N_2(e)=\{w\in V(G)\mid d(v,w)<d(u,w)\}$; and $N_0(e)=\{w\in V(G)\mid d(u,w)=d(v,w)\}$. Also, for $s\in\{0,1,2\}$, $n_s(e)=|N_s(e)|$.
The Szeged index of graph G is denoted by $Sz(G)$ and is defined as

$$Sz(G)=\sum_{e\in E(G)} n_1(e)n_2(e).$$

The Szeged index of a graph G is a topological index related to the Wiener index. Which is introduced by Iván Gutman [3]. Simić, S., Gutman, I., and Baltić, V. [4] proved that for every connected graph G(with

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at least two vertices), $Sz(G) \geq W(G)$, with equality holds if and only if each block of G is a complete graph. A block of a graph G is a maximal connected subgraph of a graph G that has no articulation/cut vertex. The trees with minimal and maximal Szeged indices are precisely those which have minimal and maximal Wiener indices. Theorem related to minimal and maximal Szeged index is stated in [5] which is as follows: Let $K_{1,n-1}$ and P_n be the n −vertex star and path, respectively and T_n is a tree other than $K_{1,n-1}$ and P_n, then $Sz(K_{1,n-1}) < Sz(T_n) < Sz(P_n)$.

Definition 7. Revised Szeged index of a graph G is denoted by $Sz^*(G)$ and defined as

$$Sz^*(G) = \sum_{e \in E(G)} \left(n_1(e) + \frac{n_0(e)}{2}\right)\left(n_2(e) + \frac{n_0(e)}{2}\right).$$

In 2002, Randić [6] proposed modification of the Szeged index and called the resulting index as a revised Szeged index. M. Aouchiche and P. Hansen [7] proved that if G is connected graph on n vertices with m edges. Then $Sz^*(G) \leq \frac{n^2 m}{4}$.

Main Results

Total graph of ladder graph: $T(L_n)$

Throughout this paper we will consider $T(L_n)$ as follows:

- Vertex set $V(T(L_n)) = V_1(T(L_n)) \cup V_2(T(L_n)) \cup V_3(T(L_n))$ where, $V_1(T(L_n)) = \{1,2,3 \cdots, 2n\}$, $V_2(T(L_n)) = \{2n+1, 2n+2, \cdots, 4n-2\}$ and $V_3(T(L_n)) = \{4n-1, 4n, \cdots, 5n-2\}$.
- Edge set $E(T(L_n)) = \cup_{i=1}^{8} E_i(T(L_n))$, where each E_i defined as follows:
  - $E_1(T(L_n)) = \{(x, x+n) | 1 \leq x \leq n\}$,
  - $E_2(T(L_n)) = \{(x, x+1), (n+x, n+x+1) | 1 \leq x \leq n-1\}$,
  - $E_3(T(L_n)) = \{(x, 4n+x-2), (n+x, 4n+x-2) | 1 \leq x \leq n\}$,
  - $E_4(T(L_n)) = \{(2n+x, 4n+x-2), (3n+x-1, 4n+x-2) | 1 \leq x \leq n-1\}$,
  - $E_5(T(L_n)) = \{(2n+x, 4n+x-1), (3n+x-1, 4n+x-1) | 1 \leq x \leq n-1\}$.
  - $E_6(T(L_n)) = \{(2n+x, 2n+x+1), (3n+x-1, 3n+x) | 1 \leq x \leq n-2\}$ and
  - $E_7(T(L_n)) = \{(2n+x, x), (3n+x-1, x+n) | 1 \leq x \leq n-1\}$.
  - $E_8(T(L_n)) = \{(2n+x, x+1), (3n+x-1, x+n+1) | 1 \leq x \leq n-1\}$.

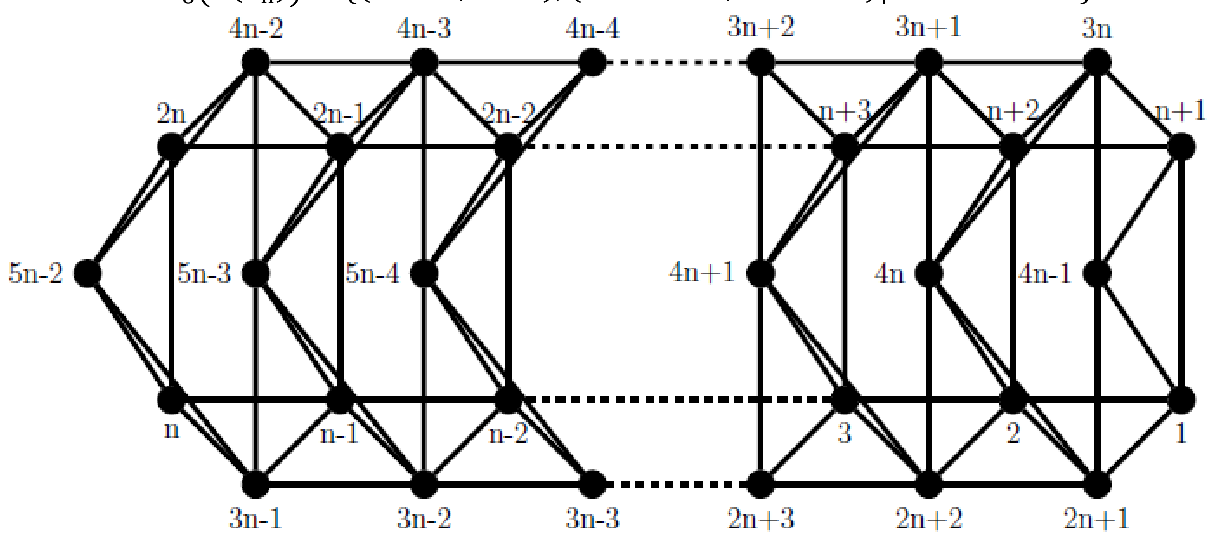

Figure 1: $T(L_n)$

Lemma 1. Let $T(L_n)$ be the total graph of ladder graph as described above then

1. $d(x, x+m) = m = d(n+x, n+x+m)$ for $x = 1,2,\dots,n-1$ and $m = 1,2,\dots,n-x$.
2. $d(n+x, 3n-1+y) = d(x, 2n+y) = \begin{cases} y-x+1 & \text{for } y \geq x \\ x-y & \text{for } y < x \end{cases}$
   for $x = 1,2,\dots,n$ and $y = 1,2,\dots,n-1$
3. $d(x, n+y) = |y-x|+1$
4. $d(x, 3n-1+y) = \begin{cases} d(3n-1+y, n+y) + d(n+y, x) & \text{for } x \leq y \\ d(3n-1+y, n+y+1) + d(n+y+1, x) & \text{for } x > y \end{cases}$
5. $d(n+x, 2n+y) = d(n+x, x) + d(x, 2n+y)$
6. $d(x, 4n-2+y) = |x-y|+1 = d(n+x, 4n-2+y)$
7. $d(3n-1+x, 4n-2+y) = d(2n+x, 4n-2+y) = \begin{cases} y-x & \text{for } x < y \\ x-y+1 & \text{for } x \geq y \end{cases}$
8. $d(2n+x, 3n-1+y) = \begin{cases} d(2n+x, 2n-1+y) + 2 & \text{for } y \geq x \\ d(3n-1+y, 3n-2+x) + 2 & \text{for } y < x \end{cases}$.

Proof. Consider n blocks of $T(L_n)$. For $2 \leq i \leq n-2$, i^{th} block contains five vertices $\{i, n+i, 2n+i, 4n-2+i, 3n-1+i\}$, 1^{st} and $(n-1)^{th}$ block contains four vertices $\{1, n+1, 2n+1, 4n-2+1, 3n-1+1\}$ and three vertices $\{n-1, 2n, 5n-2\}$ respectively. Each block contains edges of $T(L_n)$ whose both ends lie in the vertex set of respective blocks. Now edges between i^{th} and $(i+1)^{th}$ block are $(i, i+1), (n+i, n+i+1), (2n+i, i+1), (2n+i, 2n+i+1), (3n-1+i, n+i+1), (3n-1+i, 3n-1+i+1), (2n+i, 4n-2+i+1), (3n-1+i, 4n-2+i+1)$. Also there are no direct edges from i^{th} block vertices to j^{th} block vertices if $|i-j| > 1$. Now consider a graph G as $V(G) = \{B_1, B_2, \dots, B_n\}$ where B_i is a block as described above for all $1 \leq i \leq n$. B_i adjacent to B_j in G if and only if there exists $v_i \in B_i$ and $v_j \in B_j$ such that v_i is adjacent to v_j in $T(L_n)$.From this we can see that G is a path graph on n vertices.
Therefore if $1 \leq i < j \leq n$ then shortest path from B_i to B_j in G is specifically $B_i, B_{i+1}, \dots B_{j-i}, B_j$

□

Theorem 1. Let L_n be the ladder graph and $T(L_n)$ be the total graph of ladder graph then

$$Sz\big(T(L_n)\big) = \frac{1}{3}(92n^3 - 171n^2 + 178n - 96).$$

Proof. Let $e = (x, n+x) \in E_1\big(T(L_n)\big)$ for some $x \in \{1,2,\dots,n\}$. From Lemma 1 we have, $N_1(e) = \{1,2,\dots n, 2n+1, 2n+2, \dots, 3n-1\}$, $N_2(e) = \{n+1, n+2, \dots, 2n, 3n, 3n+1 \dots, 4n-2\}$ and $N_0(e) = \{4n-1, 4n4n+1, \cdots, 5n-2\}$. Therefore $n_1(e) = 2n-1 = n_2(e)$. Therefore

$$\sum_{e \in E_1(T(L_n))} n_1(e) n_2(e) = n(2n-1)^2 \tag{1}$$

Let $e = (x, x+1)$ or $e = (n+x, n+x+1)$ from $E_2\big(T(L_n)\big)$ for some $x \in \{1,2,\dots,n-1\}$. From 1 we have,

$$\sum_{e \in E_2(T(L_n))} n_1(e) n_2(e) = \sum_{k=0}^{n-2} (5k+3)\big(5\big(n-(k+1)\big)-2\big).$$

Therefore

$$\sum_{e \in E_2(T(L_n))} n_1(e) n_2(e) = \frac{(n-1)(25n^2-35n+24)}{3} \tag{2}$$

Let $e = (x, 4n+x-2)$ or $e = (n+x, 4n+x-2)$ from $E_3\big(T(L_n)\big)$ for some $x \in \{1,2,\dots,n\}$. From Lemma 1 we have, $N_1(e) = \{1,2,\cdots,n\}$, $N_2(e) = \{3n, 3n+1, \cdots, 4n-2\} \cup \{4n+x-2\}$ and $N_0(e) =$

$\{2n+1, 2n+2, \cdots, 3n-1\} \cup (\{4n-1, 4n, \cdots, 5n-2\} - \{4n+x-2\}) \cup \{n+1, n+2, \cdots 2n\}$.
$n_1(e) = n = n_2(e)$. Therefore

$$\sum_{e \in E_3(T(L_n))} n_1(e) n_2(e) = 2n^3 \tag{3}$$

Let $e = (2n+x, 4n+x-2)$ or $e = (3n+x-1, 4n+x-2)$ from $E_4(T(L_n))$ for some $z \leq x \leq n-1$. For $e = (2n+x. 4n+x-2)$, From Lemma 1 we have, $N_1(e) = \{2n+x, 2n+x+1, \cdots, 3n-1\} \cup \{4n+x-1, 4n+x, \cdots, 5n-2\} \cup \{x+1, x+2, \cdots, n\}$, $N_2(e) = \{4n+x-2\} \cup \{3n, 3n+1, \cdots, 3n-1+x\} \cup \{n+1, n+2, \cdots, n+x\}$ and $N_0(e) = \{1, 2, \cdots, x\} \cup \{2n+1, 2n+2, \cdots 2n+x\} - \{2n+x\} \cup \{n+1+x, n+x+2, \cdots, 2n\} \cup \{3n+x, 3n+x+1, \cdots, 4n-2\} \cup \{4n-1, 4n, \cdots 4n+x-2\} - \{4n+x-2\}$. Therefore

$$\sum_{e \in E_4(T(L_n))} n_1(e) n_2(e) = 2 \sum_{i=1}^{n-1} 3\big((2n-1)k - 2k^2 + n\big) = n(n-1)(2n+5) \tag{4}$$

Let $e = (2n+x, 4n+x-1)$ or $e = (3n+x-1, 4n+x-1)$ from $E_5(T(L_n))$ for some $1 \leq x \leq n-1$. Which is similar as above therefore

$$\sum_{e \in E_5(T(L_n))} n_1(e) n_2(e) = n(n-1)(2n+5) \tag{5}$$

Let $e = (2n+x, 2n+x+1)$ or $e = (3n+x-1, 3n+x)$ from $E_6(T(L_n))$ for some $1 \leq x \leq n-2$.For $e = (2n+x, 2n+x+1)$, From Lemma 1 we have, $N_1(e) = \{2n+1, 2n+2, \cdots, 2n+x\} \cup \{1, 2, \cdots, x\} \cup \{n+1. n+2, \cdots, n+x\} \cup \{3n+x-1, 3n+x-2, \cdots, 3n\} - \{3n+x-1\} \cup \{4n-1, 4n, \cdots, 4n+x-2\}$, $N_2(e) = \{2n+x+1, 2n+x+2, \cdots, 3n-1\} \cup \{x+2, x+3, \cdots, n\} \cup \{n+x+2, n+x+3, \cdots, 2n\} \cup \{3n+x, 3n+x+1, \cdots, 4n-2\} - \{3n+x\} \cup \{4n+x, 4n+x+1, \cdots 5n-2\}$ and $N_0(e) = \{x+1, 4n+x-1, n+x+1, 3n+x-1, 3n+x\}$. Therefore

$$\sum_{e \in E_6(T(L_n))} n_1(e) n_2(e) = \sum_{k=1}^{n-2} (25(n-1)k - 25k^2 + 6) = \frac{(n-2)(25n^2 - 55n + 36)}{3} \tag{6}$$

Let $e = (x, 2n+x)$ or $e = (x+n, 3n+x-1)$ from $E_7(T(L_n))$ for some $1 \leq x \leq n-1$.For $e = (x, 2n+x)$, From Lemma 1 we have, $N_1(e) = \{1, 2, \cdots, x\} \cup \{n+1, n+2, \cdots, n+x\}$, $N_2(e) = \{2n+x, 2n+x+1, \cdots, 3n-1\} \cup \{4n-1+x, \cdots, 5n-2\} \cup \{3n+x, 3n+x+1, \cdots, 4n-2\}$ and $N_0(e) = \{x+1, x+2, \cdots n\} \cup \{4n-1, \cdots 4n+x-2\} \cup \{n+x+1, \cdots 2n\} \cup \{3n, 3n+1, \cdots, 3n+x-1\} \cup \{2n+1, 2n+2, \cdots, 2n+x\} - \{2n+x\}$. Therefore

$$\sum_{e \in E_7(T(L_n))} n_1(e) n_2(e) = 2 \sum_{k=1}^{n-1} (2k(3n-1) - 6k^2) = 2n^2(n-1) \tag{7}$$

Let $e = (2n+x, x+1)$ or $e = (n+x+1, 3n+x-1)$ from $E_8(T(L_n))$ for some $1 \leq x \leq n-1$. Which is similar as above therefore

$$\sum_{e \in E_8(T(L_n))} n_1(e) n_2(e) = 2n^2(n-1) \tag{8}$$

Therefore, after adding up these 8 equations, we get,

$$Sz\big(T(L_n)\big) = \frac{92n^3 - 171n^2 + 178 - 96}{3}.$$

□

Corollary 1. Let L_n be the ladder graph and $T(L_n)$ be the total graph of ladder graph then

$$Sz^*(T(L_n)) = \frac{(n-1)(149n^2-61n-26)}{2}.$$

Proof. For $E_1(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_1(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{n}{4}(5n-2)^2. \tag{1}$$

For $E_2(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_2(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-1)(25n^2-5n+6)}{3}. \tag{2}$$

For $E_3(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_3(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{n}{4}(5n-2)^2. \tag{3}$$

For $E_4(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_4(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-1)(136n^2-89n+18)}{12} \tag{4}$$

For $E_5(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_5(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-1)(136n^2-89n+18)}{12} \tag{5}$$

For $E_6(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_6(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-2)(50n^2+40n-63)}{6}. \tag{6}$$

For $E_7(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_7(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-1)(136n^2-89n+18)}{12} \tag{7}$$

For $E_8(T(L_n))$, from Theorem 1 we have,

$$\sum_{e\in E_8(T(L_n))}\left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(n-1)(136n^2-89n+18)}{12} \tag{8}$$

Therefore, after adding up 8 equations, we get,

$$Sz^*(T(L_n)) = \frac{(n-1)(149n^2-61n-26)}{2}.$$

□

Total graph of wheel graph: $T(W_{n+1})$

Let W_{n+1} be the wheel graph with $n+1$ vertices out of these $n+1$ vertices 1 vertex is apex vertex and other n vertices are on rim. We labeled $n+1$ vertices $V(W_{n+1}) = \{1,2,\dots n+1\}$ as apex vertex labeled $n+1$ and rim vertices labeled in such a way that $E(W_{n+1}) = \{(n+1,x) | 1 \le x \le n\} \cup \{(x, x+1),(n,1) | 1 \le x \le n-1\}$.

Total graph of wheel graph $T(W_{n+1})$ has $3n+1$ vertices. Consider a partition of these $3n+1$ vertices into 4 non-empty disjoint sets as $V_1(T(W_{n+1})) = \{1,2,\dots,n\}$, $V_2(T(W_{n+1})) = \{n+1\}$, $V_3(T(W_{n+1})) =$

$\{n+2, n+3 \ldots, 2n+1\}$ and $V_4(T(W_{n+1})) = \{2n+2, 2n+3, \ldots, 3n+1\}$. Therefore $V(T(W_{n+1})) = V_1 \cup V_2 \cup V_3 \cup V_4$. Consider a partition of edge set $E(T(W_{n+1}))$ of $T(W_{n+1})$ into 7 non-empty disjoints sets as $E(T(W_{n+1})) = \cup^6{}_{i=1} E_i(T(W_{n+1}))$. Where

- $E_1(T(W_{n+1})) = \{(n+1, x) : 1 \le x \le n\}$
- $E_2(T(W_{n+1})) = \{(x, x+1), (n, 1) : 1 \le x \le n-1\}$
- $E_3(T(W_{n+1})) = \{(x, n+1+x), (n+1+x, x+1), (n, 2n+1), (2n+1, 1) : 1 \le x \le n-1\}$
- $E_4(T(W_{n+1})) = \{(x, 2n+x+1), (2n+x+1, n+1) : 1 \le x \le n\}$
- $E_5(T(W_{n+1})) = \{(n+x+1, 2n+x+1), (n+x+1, 2n+1+x+1) : 1 \le x \le n\}$, where $3n+2 = 2n+2$
- $E_6(T(W_{n+1})) = \{(2n+x, 2n+y) : 2 \le x, y \le n+1, x \ne y\}$
- $E_7(T(W_{n+1})) = \{(n+1+x, n+1+x+1) | 1 \le x \le n\}$

Lemma 2. Let $T(W_{n+1})$ be the graph as described above then

1. For $x \in V_1(T(W_{n+1}))$, $d(x, n+1) = 1$.
2. For $x, y \in V_1(T(W_{n+1}))$, $d(x, y) = \{1 \text{ for } |x-y| = 1 \text{ or } n-1 \; 2 \text{ for } |x-y| \ge 2$.
3. For $x, y \in \{1, 2, \cdots n\}$, $d(x, 2n+y) = \{1 \text{ for } x = y \; 2 \text{ for } x \ne y$
4. For $x \in V_4(T(W_n+1))$, $d(x, n+1) = 1$.
5. For $x, y \in V_4(T(W_{n+1}))$, $d(x, y) = 1$.
6. For $x, y \in \{2, 3, \cdots n+1\}$, $d(n+x, 2n+y) = \{1 \text{ for } x = y \; 2 \text{ for } x \ne y$
7. For $x \in V_3(T(W_{n+1}))$, $d(x, n+1) = 2$.
8. For $1 \le x, y \le n$,
$$d(x, n+1+y) = \begin{cases} 1 & \text{for } |x-y| = 0, 1, n-1 \text{ and } (x, y) \ne (n, 1) \\ 2 & \text{for } |x-y| = 2, n-2, \text{ and } (x, y) = (n, 1), (x, y) \ne (n, 2) \\ 3 & \text{for } 3 \le |x-y| < n-2, \text{ and } (x, y) = (n, 2) \end{cases}$$
9. For $1 \le x, y \le n$,
$$d(n+1+x, n+1+y) = \begin{cases} 1 & \text{for } |x-y| = 1, n-1 \\ 2 & \text{for } |x-y| = 2, n-2 \\ 3 & \text{for } 3 \le |x-y| < n-2 \end{cases}$$

Theorem 2. Let $T(W_{n+1})$ be the total graph of wheel graph W_{n+1} then
$$Sz(T(W_{n+1})) = n(52n-49).$$

Proof. For edges from set $E_1(T(W_{n+1}))$, $e = (n+1, x)$. From Lemma 2 we have, $N_1(e) = \{2n+1+y : y \ne x\} \cup \{y : 1 < |y-x| < n-1\} \cup \{n+1+y : 2 < |y-x| < n-2\} \cup \{n+1\}$, $N_2(e) = \{n+1+y : y = x \text{ or} y = x-1\} \cup \{x\}$ and $N_0(e) = \{x+1, x-1, n+1+x+1, n+1+x-2, 2n+1+x\}$ $|N_0(e)| = n_0(e) = 5$. Therefore $|N_1(e)| = n_1(e) = 3n-7$ $|N_2(e)| = n_2(e) = 3$. Therefore,
$$\sum_{e \in E_1(T(W_{n+1}))} n_1(e) n_2(e) = 3n(3n-7) \tag{1}$$
For edges from set $E_2(T(W_{n+1}))$. From Lemma 2 we have, $N_1(e) = \{x, x-1, 2n+1+x, n+1+x-1, n+1+x-2\}$, $N_2(e) = \{x+1, x+2, 2n+1+x+1, n+1+x+1, n+1+x+2\}$ and $N_0(e) = \{y : 2 \le |y-x| \le n-2\} \cup \{2n+1+y : y \ne x, y \ne x+1\}$ $\cup \{n+1+y : y \ne x \pm 1, y \ne x \pm 2\} \cup \{n+1\}$. Therefore $|N_1(e)| = n_1(e) = 5 = n_2(e) = |N_2(e)|$. Therefore,

$$\sum_{e\in E_2(T(W_{n+1}))} n_1(e)n_2(e) = 25n \tag{2}$$

For edges from set $E_3(T(W_{n+1}))$. For $e = (x, n+1+x)$, from Lemma 2 we have, $N_1(e) = \{y: y \neq x+1, y \neq x+2\}$, $N_2(e) = \{2n+1+x+1, n+1+x, n+1+x+1, n+1+x+2\}$ and $N_0(e) = \{x+1, x+2\} \cup \{2n+1+y: y \neq x\} \cup \{n+1+y: y \neq x, y \neq x+1, y \neq x+2\}$. Therefore $|N_1(e)| = n_1(e) = n-1$. $|N_2(e)| = n_2(e) = 4$. For $e = (x+1, n+1+x), (n, 2n+1), (1, 2n+1)$, from Lemma 2 we have, $N_1(e) = \{y: y \neq x, y \neq x-1\} \cup \{n+1\}$, $N_2(e) = \{2n+1+x, n+1+x, n+1+x-1, n+1+x-2\}$ and $N_0(e) = \{x-1, x-2\} \cup \{2n+1+y: y \neq x+1\} \cup \{n+1+y: y \neq x, y \neq x-1, y \neq x-2\}$. Therefore $|N_1(e)| = n_1(e) = n-1$, $|N_2(e)| = n_2(e) = 4$. Therefore

$$\sum_{e\in E_3(T(W_{n+1}))} n_1(e)n_2(e) = 8n(n-1) \tag{3}$$

For edges from set $E_4(T(W_{n+1}))$, from Lemma 2 we have, $N_1(e) = \{x, x+1, x-1\}$, $N_2(e) = \{2n+1+y: 1 \leq y \leq n\} \cup \{n+1+y: y \neq x, y \neq x \pm 1, y \neq x-2\}$ and $N_0(e) = \{y: y \neq x, y \neq x \pm 1\} \cup \{n+1+x, n+1+x-1, n+1+x+1, n+1+x-2, n+1\}$. Therefore $|N_1(e)| = n_1(e) = 3$ and $|N_2(e)| = n_2(e) = 2n-4$. Therefore, $n_1(e)n_2(e) = 6(n-2)$. For $e = (2n+x+1, n+1)$ from Lemma 2 we have,$N_1(e) = \{2n+x+1, n+1+x, n+1+x+1\}$, $N_2(e) = \{y: y \neq x\} \cup \{n+1\}$ and $N_0(e) = \{2n+1+y: y \neq x\} \cup \{n+1+y: y \neq x, y \neq x-1\} \cup \{x\}$. Therefore $|N_1(e)| = n_1(e) = 3$ and$|N_2(e)| = n_2(e) = n$. Therefore,

$$\sum_{e\in E_4(T(W_{n+1}))} n_1(e)n_2(e) = 3n(3n-4) \tag{4}$$

For edges from set $E_5(T(W_{n+1}))$. For $e = (n+1+x, 2n+1+x)$, from Lemma 2 we have $N_1(e) = \{n+1+x, n+1+x+1, x+1\}$, $N_2(e) = \{n+1+y: y \neq x, y \neq x \pm 1, y \neq x \pm 2\} \cup \{y: y \neq x, y \neq x \pm 1, y \neq x+2\} \cup \{n+1\}$ and $N_0(e) = \{n+1+x-1, n+1+x-2, n+1+x+2, x, x+2, x-1, 2n+1+x+1\}$. Therefore $|N_1(e)| = n_1(e) = 3$ and $|N_2(e)| = n_2(e) = 3n-9$. For $e = (n+1+x, 2n+1+x+1)$, from Lemma 2 we have, $N_1(e) = \{n+1+x, x+1+x-1, x\}$,$N_2(e) = \{n+1+y: y \neq x, y \neq x \pm 1, y \neq x \pm 2\} \cup \{y: y \neq x, y \neq x \pm 1, y \neq x+2\} \cup \{2n+1+y: y \neq x\} \cup \{n+1\}$ and $N_0(e) = \{x+1, n+1+x+1, 2n+1+x, n+1+x+2, n+1+x-2, x+2, x-1\}$. Therefore $|N_1(e)| = n_1(e) = 3$ and $|N_2(e)| = n_2(e) = 3n-9$. Therefore,

$$\sum_{e\in E_5(T(W_{n+1}))} n_1(e)n_2(e) = 18n(n-3) \tag{5}$$

For edges from set $E_6(T(W_{n+1}))$. For $e = (2n+x, 2n+y)$, where $|x-y| = 1$ or $n-1$, from Lemma 2 we have $N_1(e) = \{x, n+1+x-1, 2n+1+x\}$, $N_2(e) = \{2n+1+x+1, x+1, n+1+x+1\}$ and $N_0(e) = \{2n+1+y, y: y \neq x, y \neq x+1\} \cup \{n+1+y: y \neq x \pm 1\} \cup \{n+1\}$. Therefore, $|N_1(e)| = n_1(e) = 3 = n_2(e) = |N_2(e)|$. For $e = (2n+1+x, 2n+1+z)$ where $\leq 2|x-z| < n-1$, from Lemma 2 we have, $N_1(e) = \{n+1+x, x, n+1+x-1, 2n+1+x\}$, $N_2(e) = \{2n+1+z, z, n = 1+z, n+1+z-1\}$ and $N_0(e) = \{y, 2n+1+y: y \neq x, y \neq z\} \cup \{n+1+y: y \neq x, y \neq z, y \neq x-1, y \neq z-1\} \cup \{n+1\}$. Therefore, $|N_1(e)| = n_1(e) = 4 = n_2(e) = |N_2(e)|$. Therefore,

$$\sum_{e\in E_6(T(W_{n+1}))} n_1(e)n_2(e) = n(8n-15) \tag{6}$$

For edges from set $E_7(T(W_{n+1}))$, from Lemma 2 we have, $N_1(e) = \{n+1+x, x+1+x-1, n+1+x-2, x, x-1, 2n+1+x\}$, and $N_0(e) = \{n+1+y: y \neq x, y \neq x \pm 1, y \neq x \pm 2, y \neq x+3\} \cup \{y: y \neq x, y \neq x-1, y \neq x+2, y \neq x+3\} \cup \{2n+1+y: y \neq x, y \neq x+2\} \cup \{n+1\}$. Therefore, $|N_1(e)| = n_1(e) = 6 = n_2(e) = |N_2(e)|$. Therefore,

$$\sum_{e\in E_7(T(W_{n+1}))} n_1(e)n_2(e) = 36n \tag{7}$$

Therefore, after adding up all 7 equations, we get,

$$Sz(T(W_{n+1})) = n(52n-49).$$

□

Corollary 2. Let W_{n+1} be the wheel graph and $T(W_{n+1})$ be the total graph of wheel graph then

$$Sz^*(T(W_{n+1})) = \frac{n(9n^3+99n^2+713n-915)}{8}.$$

Proof. For $E_1(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_1(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{33n}{4}(2n-3) \tag{1}$$

For $E_2(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_2(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(3n+1)^2n}{4} \tag{2}$$

For $E_3(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_3(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = 4n(n-1)(n+3) \tag{3}$$

For $E_4(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_4(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{n}{4}(17n^2+93n-6n) \tag{4}$$

For $E_5(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_5(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{13n}{2}(6n-11) \tag{5}$$

For $E_6(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_6(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(3n+1)^2n}{4} + \frac{(3n+1)^2n(n-3)}{8} \tag{6}$$

For $E_7(T(W_{n+1}))$, from Theorem 2 we get,

$$\sum_{e\in E_7(T(W_{n+1}))} \left(n_1(e)+\frac{n_0(e)}{2}\right)\left(n_2(e)+\frac{n_0(e)}{2}\right) = \frac{(3n+1)^2n}{4} \tag{7}$$

Therefore, after adding up all 7 equations, we get,

$$Sz^*(T(W_{n+1})) = \frac{n(9n^3+99n^2+713n-915)}{8}.$$

□

Concluding Remarks and Future Scope

In this paper we have derived formulae of Szeged index and Revised Szeged index of $T(L_n)$ and $T(W_{n+1})$. One can obtain formulas for the total graph of Mobius ladder graph, triangular ladder graph and for similar graphs.

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